The final approach to steady, viscous flow near a stagnation point following a change in free stream velocity

By R. E. KELLY

Aerodynamics Division, National Physical Laboratory, Teddington, Middlesex†

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The flow field near a stagnation point in two-dimensional, incompressible, viscous flow is considered to change with time in such a way that the inviscid flow is steady after some given finite instant of time. The final approach to steady flow throughout the field is shown to be characterized by exponential decay with time of perturbations from the steady velocity field. The characteristic factors in the exponents arise from the solution of an eigenvalue problem in ordinary linear differential equations.

Similar behaviour exists for the axially symmetric case. A comparable analysis furnishes, however, a meaningless result in the case of a two-dimensional, semi-infinite flat plate which is moving in its own plane, normal to its leading edge.

1. Introduction

Unsteady viscous flow investigations to date fall generally into one of two classes of problem. In the first class, the free stream velocity changes at every instant of time, often in a periodic manner (e.g. Lighthill 1954). In the second class, the body accelerates, often impulsively, from rest (e.g. Stewartson 1951).

The present problem concerns incompressible flow near a stagnation point for a variation with time of the free stream such that the inviscid flow is steady after some given finite instant of time. An investigation is made of the development of steady flow in the viscous region after this given instant. From this limited point of view, it is unnecessary to define the original imposed unsteadiness further, other than to assume that it allows the flow eventually to approach the standard steady-state form. As such, the analysis may be viewed as a first step towards determining the unsteady stagnation flow caused by a continuous disturbance acting during a finite time interval, such as Watson (1958) treated for the case of flow past an infinite plane porous wall. Further work might attempt to match the present limiting behaviour to any solution valid for earlier times. The problem therefore lies between the two main groups of work.

Perturbations from the steady velocity field are shown to decay exponentially with time, with the characteristic factors in the exponents appearing as eigenvalues. Such behaviour has already been noted by Rott & Rosenzweig (1960) for the case of a two-dimensional stagnation flow when the wall moves im-

† Now at Department of Aeronautics and Astronautics, M.I.T.

pulsively in its own plane. This constitutes a somewhat simpler problem because the velocity normal to the wall may be shown to be independent of time, and the governing equation for the velocity parallel to the wall is linear. The eigenvalues were not determined in their analysis.

2. Two-dimensional stagnation-point flow

Let x denote the distance along the wall, z the distance normal to it, u and w the corresponding components of velocity, t the time, p the pressure, ρ the density, and ν the kinematic viscosity. For incompressible flow, the Navier–Stokes equations and the equation of continuity are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2.1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0. \tag{2.3}$$

The boundary conditions are

$$u = w = 0 \quad \text{at} \quad z = 0, \quad u \to U(x,t) \quad \text{as} \quad z \to \infty. \tag{2.4}$$

The initial conditions on t are discussed in § 2.2. A similarity solution is now sought which involves the usual linear dependence on x. The following substitutions are made

$$\left.\begin{aligned} \bar{\tau} = at, \quad \zeta = z[ag(\bar{\tau})/\nu]^{\frac{1}{2}}, \quad U(x,t) = axg(\bar{\tau}),\\ u = axg(\bar{\tau})\,\phi'(\zeta,\bar{\tau}), \quad w = -[a\nu g(\bar{\tau})]^{\frac{1}{2}}\,\phi(\zeta,\bar{\tau}),\\ p = -\tfrac{1}{2}\rho[a^2x^2g^2(\bar{\tau})+a\nu G(\zeta,\bar{\tau})]-\tfrac{1}{2}\rho a^2x^2\dot{g}(\bar{\tau}), \end{aligned}\right\} \tag{2.5}$$

where primes denote differentiation with respect to ζ and dots denote differentiation with respect to $\bar{\tau}$. The Navier–Stokes equations are reduced to the form

$$\dot{g}\phi'+g\dot{\phi}'+\tfrac{1}{2}\zeta\dot{g}\phi''+(g\phi')^2-g^2\phi\phi'' = g^2+\dot{g}+g^2\phi''', \tag{2.6}$$

$$-\tfrac{1}{2}\dot{g}\phi-g\dot{\phi}-\tfrac{1}{2}\zeta\dot{g}\phi'+g^2\phi\phi' = \tfrac{1}{2}gG'-g^2\phi'', \tag{2.7}$$

with the boundary conditions

$$\phi(0,\bar{\tau}) = \phi'(0,\bar{\tau}) = 0, \quad \phi'(\infty,\bar{\tau}) = 1. \tag{2.8}$$

Equation (2.7) merely determines the pressure function, G, once ϕ is known from (2.6) and may therefore be omitted from further consideration. No restriction has yet been placed on $g(\bar{\tau})$. For instance, if $g(\bar{\tau})$ varies slowly with time, $\phi(\zeta,\bar{\tau})$ may be expanded in terms of g and its derivatives to yield the same set of equations as Moore (1957) obtained for unsteady Falkner–Skan flow, once his results are specialized to the stagnation case. The function $g(\bar{\tau})$ is now restricted to one which reaches a constant value, g_1, at some finite time and remains constant thereafter. Then equation (2.6) reduces to

$$\phi'''+\phi\phi''-(\phi')^2+1-(g_1)^{-1}\dot{\phi}' = 0. \tag{2.9}$$

With $\tau = g_1\bar{\tau}$, (2.9) becomes

$$\phi''' + \phi\phi'' - (\phi')^2 + 1 - \partial\phi'/\partial\tau = 0. \tag{2.10}$$

A solution is now sought which represents a perturbation from the steady state. Therefore, let

$$\phi(\zeta, \tau) = \phi_0(\zeta) + \epsilon\phi_1(\zeta, \tau), \tag{2.11}$$

where $\phi_0(\zeta)$ is the steady solution and ϵ represents a small parameter. After substitution into (2.10), the resulting equations are

$$\phi_0''' + \phi_0\phi_0'' - (\phi_0')^2 + 1 = 0, \tag{2.12}$$

$$\phi_1''' + \phi_0\phi_1'' - 2\phi_0'\phi_1' + \phi_0''\phi_1 - \partial\phi_1'/\partial\tau = 0. \tag{2.13}$$

The validity of linearization will be discussed in § 2.2. The boundary conditions are

$$\left.\begin{aligned} \phi_0(0) = \phi_0'(0) = 0, \quad \phi_0'(\infty) = 1, \\ \phi_1(0,\tau) = \phi_1'(0,\tau) = 0, \quad \phi_1'(\infty,\tau) = 0. \end{aligned}\right\} \tag{2.14}$$

Initial conditions with regard to time define the particular 'small-time' solution to which the present 'large-time' solution may be joined and will therefore remain unspecified. They will be discussed further in § 2.2. Accurate values of ϕ_0 are available from several sources, e.g. Schlichting (1955, p. 73).

The perturbation is now assumed to be separable in the form

$$\phi_1(\zeta, \tau) = \Phi(\zeta)K(\tau). \tag{2.15}$$

The validity of this assumption is discussed in § 2.2.

Upon substitution of (2.15) into (2.13), it becomes immediately evident that $K(\tau)$ must be of the form

$$K(\tau) = e^{-\lambda\tau}, \tag{2.16}$$

where $(-\lambda)$ is a constant. Therefore $\Phi(\zeta)$ satisfies the equation

$$\Phi''' + \phi_0\Phi'' - 2\phi_0'\Phi' + \phi_0''\Phi + \lambda\Phi' = 0, \tag{2.17}$$

$$\Phi(0) = \Phi'(0) = 0, \quad \Phi'(\infty) = 0. \tag{2.18}$$

Because the external flow is time-independent we expect the time-dependent perturbations to be confined to a boundary layer, in which a time lag due to viscosity is present. Thus we expect that $\Phi' \to 0$ exponentially as $\zeta \to \infty$. For some ranges of λ we shall find that solutions, if they exist, *must* have this behaviour. For other ranges of λ, however, it is necessary to exclude solutions for which $\Phi' \to 0$ algebraically as $\zeta \to \infty$ and to retain only the exponentially decaying solution. The eigenvalue, λ, will be seen to have a discrete spectrum with these conditions.

One solution of (2.17) is

$$\Phi = \phi_0' - \lambda, \tag{2.19}$$

and the order of the equation may therefore be reduced by the substitution

$$\Phi(\zeta) = (\phi_0' - \lambda)\theta(\zeta). \tag{2.20}$$

The reduced equation is

$$\theta''' + \left[3\frac{\phi_0''}{\phi_0' - \lambda} + \phi_0\right]\theta'' + \left[\frac{3\phi_0''' + 2\phi_0\phi_0''}{\phi_0' - \lambda} + (\lambda - 2\phi_0')\right]\theta' = 0. \tag{2.21}$$

If we assume that $\lambda \neq 0$ or 1, the boundary conditions (2.18) become

$$\theta(0) = \theta'(0) = 0, \quad \theta'(\infty) = 0. \tag{2.22}$$

(In the special case of $\lambda = 0$, the second condition (2.22) must be replaced by $\theta'(0) = 1$, the other conditions being unaltered. For $\lambda = 1$ a different condition at infinity is required.) Equation (2.21) may be simplified by the further transformation

$$\theta'(\zeta) = (\phi_0' - \lambda)^{-2}\psi(\zeta), \tag{2.23}$$

which gives

$$\psi'' + \left[\frac{-\phi_0''}{\phi_0' - \lambda} + \phi_0\right]\psi' + \left[\frac{\phi_0'''}{\phi_0' - \lambda} + \lambda - 2\phi_0'\right]\psi = 0, \tag{2.24}$$

$$\psi(0) = 0, \quad \psi(\infty) = 0. \tag{2.25}$$

(For $\lambda = 0$ we must impose $\psi'(0) = 0$ also.) Equation (2.24), subject to the homogeneous boundary conditions (2.25), poses an eigenvalue problem for the parameter λ. In the next section we seek to determine bounds on the possible regions in the complex plane which can contain λ.

2.1. *Qualitative discussion of the eigenvalue, λ*

Our aim in this section is twofold. First, we wish to show that the steady-state flow is stable with regard to small perturbations of the form (2.15). We therefore wish to prove that all possible eigenvalues must have positive real parts. This result allows us to construct a solution which describes the final approach to the steady state for at least a certain class of problems. The particular members would, in principle, be defined by the initial conditions. Secondly, we wish to indicate by a combination of mathematical and physical arguments that the existence of complex eigenvalues is unlikely.

We first consider possible real eigenvalues. Without loss in generality, we may consider the eigenfunctions to be normalized by imposing the condition $\psi'(0) = 1$. This effectively guarantees through (2.24) and (2.25) that, for any real λ, the corresponding eigenfunction will be real. Equation (2.24) is now placed in its self-adjoint form through multiplication by $(\phi_0' - \lambda)^{-1}E\psi$, where

$$E = \exp\left\{\int_0^\zeta \phi_0 \, d\zeta\right\}, \tag{2.26}$$

and by integration from $\zeta = 0$ to ∞

$$\int_0^\infty \left\{\psi \frac{d}{d\zeta}\left[\frac{E}{\phi_0' - \lambda}\psi'\right] + E\left[\frac{\phi_0'''}{(\phi_0' - \lambda)^2} + \frac{\lambda - 2\phi_0'}{\phi_0' - \lambda}\right]\psi^2\right\} d\zeta = 0. \tag{2.27}$$

Integration by parts then yields

$$\left[\frac{E}{\phi_0' - \lambda}\psi\psi'\right]_0^\infty - \int_0^\infty \left\{\frac{E}{\phi_0' - \lambda}(\psi')^2 - E\left[\frac{\phi_0'''}{(\phi_0' - \lambda)^2} + \frac{\lambda - 2\phi_0'}{\phi_0' - \lambda}\right]\psi^2\right\} d\zeta = 0. \tag{2.28}$$

It will be indicated below that the integrated term vanishes and that the integral is convergent (if real values of λ in the range $0 < \lambda < 1$ are excluded). Therefore (2.28) may be written

$$\int_0^\infty \frac{E}{\phi_0' - \lambda}\left\{(\psi')^2 - \left[\frac{\phi_0'''}{\phi_0' - \lambda} + (\lambda - 2\phi_0')\right]\psi^2\right\} d\zeta = 0. \tag{2.29}$$

Because ϕ_0''' is negative throughout and ϕ_0' is positive, the relation cannot be satisfied for $\lambda \leqslant 0$. Therefore real values of λ, if they exist, are positive. This result means that the non-oscillatory perturbations, if they exist, decay to zero as $\tau \to \infty$.

The asymptotic form of ψ is required in order to demonstrate the vanishing of the integrated term and the convergence of the integral as $\zeta \to \infty$. In this region $\phi_0 \sim \zeta - \alpha$, where α is a constant. The asymptotic solution of (2.24) may be expressed in terms of the parabolic cylinder functions (Whittaker & Watson 1958, p. 347). If $\xi = \zeta - \alpha$, then as $\xi \to \infty$

$$\psi \sim C_1 e^{-\frac{1}{2}\xi^2}(\xi)^{\lambda-3} + C_2(i\xi)^{-\lambda+2}, \tag{2.30}$$

where the asymptotic form of the above functions has been employed. This formula is the basis of the remark, made after (2.18), that it is necessary to exclude an algebraically decaying solution for certain ranges of λ. To achieve a boundary-layer solution, therefore, C_2 is equated to zero. Then, for large ξ,

$$\left.\begin{aligned} \psi &\sim \xi^{\lambda-3} e^{-\frac{1}{2}\xi^2}, \\ \psi' &\sim \xi^{\lambda-2} e^{-\frac{1}{2}\xi^2}, \\ E &\sim e^{\int \xi\, d\xi} \sim e^{\frac{1}{2}\xi^2}, \end{aligned}\right\} \tag{2.31}$$

By examining the relevant terms in (2.28) one sees that the integrated term vanishes and, because the integrand becomes exponentially small, that the integral is convergent as $\zeta \to \infty$. The above argument does not apply for $\lambda = 1$ but similar conclusions are valid. For the important case of $\lambda = 0$ it can be shown that the integral (2.28) converges at $\zeta = 0$ and that the integrated term vanishes at $\zeta = 0$, facts which are clear for other values of λ.

The existence of discrete values of λ may be demonstrated, at least for sufficiently large values of λ. Returning to equation (2.24), let

$$\psi(\zeta) = (\lambda - \phi_0')^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\int_0^\zeta \phi_0\, d\zeta\right\} \Omega(\zeta). \tag{2.32}$$

Equation (2.24) becomes

$$\Omega'' + \left[\lambda - \tfrac{5}{2}\phi_0' - \tfrac{1}{4}\phi_0^2 + \frac{3}{2}\frac{\phi_0'''}{\phi_0' - \lambda} - \frac{3}{4}\left(\frac{\phi_0''}{\phi_0' - \lambda}\right)^2 + \frac{1}{2}\frac{\phi_0\phi_0''}{\phi_0' - \lambda}\right]\Omega = 0. \tag{2.33}$$

The boundary conditions on Ω are

$$\Omega(0) = 0, \quad \Omega(\infty) = 0. \tag{2.34}$$

For large values of λ, (2.33) is, approximately,

$$\Omega'' + [\lambda - \tfrac{5}{2}\phi_0' - \tfrac{1}{4}\phi_0^2]\,\Omega = 0. \tag{2.35}$$

Because ϕ_0 and ϕ_0' are both positive, the form of (2.35) and the boundary conditions on Ω fulfil the conditions given by Titchmarsh (1946, p. 113) for the existence of a discrete set of eigenvalues, λ_j $(j = 1, 2, \ldots)$, where λ_j tends to infinity as j tends to infinity.

The eigenvalues have been assumed real up to this point. It will now be shown that complex eigenvalues, if they exist, also have positive real parts and give perturbations which decay with time. Also, by establishing certain bounds on the eigenvalues, we will see that the existence of complex eigenvalues is unlikely. The author is indebted to Dr J. T. Stuart for the following analysis.

Multiply (2.24) by $(\phi_0'-\lambda)^{-1}E\psi^*$, where ψ^* is the complex conjugate of ψ, and integrate from $\zeta = 0$ to infinity. After an integration by parts, with the assumption that the integrated term vanishes at the limits, there results

$$\int_0^\infty \left\{\frac{E}{\phi_0'-\lambda}\psi'^*\psi' - E\left[\frac{\phi_0'''}{(\phi_0'-\lambda)^2}+\frac{\lambda-2\phi_0'}{\phi_0'-\lambda}\right]\psi^*\psi\right\}d\zeta = 0. \quad (2.36)$$

If we subtract the complex conjugate, we can deduce that

$$(\lambda-\lambda^*)\int_0^\infty \frac{E}{|\phi_0'-\lambda|^2}\left\{\psi'^*\psi' + \left[\frac{2\phi_0'''(\lambda_r-\phi_0')}{|\phi_0'-\lambda|^2}+\phi_0'\right]\psi^*\psi\right\}d\zeta = 0, \quad (2.37)$$

where λ^* is the complex conjugate of λ and λ_r is the real part of λ. Because $\phi_0''' \leqslant 0$ and $\phi_0' \geqslant 0$, the integrand is positive throughout if $\lambda_r \leqslant 0$. Hence (2.37) can be satisfied only if $\lambda = \lambda^*$, i.e. complex eigenvalues with negative real parts do not exist. As in the real analysis, it can be shown that the integral (2.37) is convergent and that the integrated term of (2.36) vanishes at the limits.

Now perform similar steps with regard to (2.33), i.e. multiply by Ω^*, etc. The following relation is obtained

$$(\lambda-\lambda^*)\int_0^\infty \left\{(\tfrac{3}{2}\phi_0''' + \tfrac{1}{2}\phi_0\phi_0'')\frac{1}{|\phi_0'-\lambda|^2} - \frac{3}{2}\frac{(\phi_0'')^2(\phi_0'-\lambda_r)}{|\phi_0'-\lambda|^4}+1\right\}\Omega\Omega^* d\zeta = 0. \quad (2.38)$$

It is immediately clear that for $|\lambda| \gg 1$ the relation can only be satisfied if $\lambda = \lambda^*$. Hence there are no complex eigenvalues with very large arguments.

In order to achieve a more precise bound, we notice that for $\lambda_r \geqslant 1$ the second term of the integrand is positive throughout, and therefore seek a condition which ensures that the sum of the first and last terms of the integrand is non-negative. By use of (2.12), we see that

$$\tfrac{3}{2}\phi_0''' + \tfrac{1}{2}\phi_0\phi_0'' = \phi_0''' - \tfrac{1}{2}(1-\phi_0'^2) = -A(\zeta) \leqslant 0. \quad (2.39)$$

It can be shown that $A(\zeta)$ has a maximum value of $\frac{3}{2}$. Further, the minimum value of $|\phi_0'-\lambda|^2$ is

$$|\phi_0'-\lambda|^2_{\min} = (1-\lambda_r)^2+\lambda_i^2, \quad (2.40)$$

where λ_i is the imaginary component of λ. Hence the sum of the first and last terms of the integrand will surely be non-negative if λ satisfies the condition

$$\frac{-\frac{3}{2}}{(1-\lambda_r)^2+\lambda_i^2}+1 \geqslant 0, \quad (2.41)$$

or

$$\tfrac{3}{2} \leqslant (1-\lambda_r)^2+\lambda_i^2. \quad (2.42)$$

Thus, if $\lambda_r \geqslant 1+(\frac{3}{2})^{\frac{1}{2}} = 2{\cdot}2247$, the integrand is positive for all λ_i. If

$$1 \leqslant \lambda_r \leqslant 1+(\tfrac{3}{2})^{\frac{1}{2}},$$

the integrand is positive if $\lambda_i^2 > \frac{3}{2}-(1-\lambda_r)^2$. In these regions, therefore, (complex) eigenvalues are not possible. But there may be eigenvalues on the real axis ($\lambda_i = 0$).

In summary, there can be no eigenvalues in the shaded region of figure 1. Further analysis indicates that the boundary curve may be extended for $\lambda_r < 1$ along a curve which intersects the imaginary axis at $\lambda_i = 1{\cdot}33$. However, figure 1 serves as an indication that complex eigenvalues probably do not exist. Calculations, given below, determine the first real eigenvalue as $\lambda_1 = 3{\cdot}063$. Hence complex eigenvalues, if they exist, would predominate in describing the behaviour of the flow. While the above investigation is general, consider the special case of impulsive flow. Here we would expect the flow to behave mainly

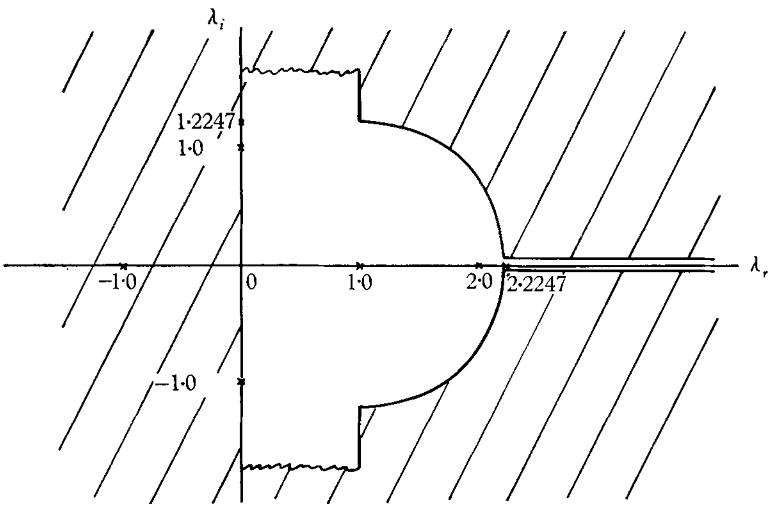

FIGURE 1. The λ-plane. No eigenvalues exist within the shaded area.

in a monotonic manner, i.e. governed largely by the lowest real eigenvalue, and any oscillatory motion to be a lesser effect. In fact, the 'small-time' solution of Goldstein & Rosenhead (1936) for impulsive flow shows no trace of oscillatory behaviour. Hence we might expect $\lambda_r > 3{\cdot}063$ which, with reference to figure 1, is impossible for complex eigenvalues. This physical reasoning, coupled with the peculiar distribution of possible complex eigenvalues, gives one some confidence that no complex eigenvalues exist.

It might also be noted that the transformations (2.20) and (2.23) become singular at some value of ζ for $0 < \lambda < 1$ because ϕ_0' ranges from zero to one. The integral (2.29) then no longer converges. While no strict proof can be offered that $\lambda > 1$, the calculation of $\lambda_1 = 3{\cdot}063$ employed the full third-order equation (2.16). Hence one need not be too concerned about the singularity.

2.2. *Determination of the eigenvalues and eigenfunctions*

While it is easy to show that (2.29) is a variational integral, which suggests use of a variational procedure to calculate the eigenvalues, inaccuracies arise of such magnitudes through approximation of the exponential factor E that the variational approach is of little use. Hence equation (2.17) will be used to calculate the eigenvalues. A Pohlhausen method will be employed to calculate the lowest eigenvalue, and then values of the first two eigenvalues and eigenfunctions, obtained by use of an electronic digital computer, will be given.

To apply a Pohlhausen technique, (2.17) is first integrated to give

$$[\Phi'']_0^\infty + [\phi_0 \Phi']_0^\infty - \int_0^\infty \phi_0' \Phi' d\zeta - 2\int_0^\infty \phi_0' \Phi' d\zeta + [\phi_0' \Phi]_0^\infty - \int_0^\infty \phi_0' \Phi' d\zeta + [\lambda \Phi]_0^\infty = 0, \tag{2.43}$$

and, by use of the boundary conditions (2.14), (2.18), (2.43), is placed in the form

$$\lambda + 1 = \{\Phi(\infty)\}^{-1}\left\{\Phi''(0) + 4\int_0^\infty \phi_0' \Phi' d\zeta\right\}. \tag{2.44}$$

We assume that ϕ_0' can be expressed approximately in the form

$$\phi_0' = 1 - \frac{b}{\gamma} e^{-\gamma\zeta} - \frac{c}{2\gamma} e^{-2\gamma\zeta}, \tag{2.45}$$

where the constants b, c, γ and a further constant of integration are chosen to satisfy

$$\left.\begin{aligned} \phi_0(0) = \phi_0'(0) = 0, \quad \phi_0''(0) = 1{\cdot}2326; \\ \phi_0(\zeta) \sim \zeta - 0{\cdot}6482 \quad (\zeta \to \infty). \end{aligned}\right\} \tag{2.46}$$

The special values in (2.46) arise from the solution of (2.12) subject to (2.14), for which the reader is referred to Schlichting (1955, p. 73). Of the two possible values of γ which arise, that one is chosen which gives the best fit to known data for various values of ζ. The resulting values of the constants are $\gamma = 1{\cdot}7799$, $b = 2{\cdot}3272$, $c = -1{\cdot}0946$. The incorrect exponential behaviour of (2.45) as $\zeta \to \infty$ is assumed to be unimportant to the evaluation of the integral in (2.44).

We now assume Φ to be of the form

$$\Phi = C_1 \beta^{-1}[\zeta e^{-\beta\zeta} + (e^{-\beta\zeta} - 1)/\beta], \tag{2.47}$$

which satisfies the boundary conditions (2.18). The incorrect exponential behaviour of (2.47) as $\zeta \to \infty$ is also assumed to be unimportant.

An additional integral relation, which is required to determine β, is obtained by multiplying (2.17) by Φ'' and integrating

$$\int_0^\infty [\Phi''\Phi''' + \phi_0(\Phi'')^2 - 2\phi_0'\Phi'\Phi'' + \phi_0''\Phi\Phi'' + \lambda\Phi'\Phi'']\, d\zeta = 0. \tag{2.48}$$

After integration by parts, (2.48) becomes

$$\int_0^\infty [\phi_0''(\Phi')^2 + \phi_0(\Phi'')^2 + \phi_0''\Phi\Phi'']\, d\zeta = \tfrac{1}{2}[\Phi''(0)]^2, \tag{2.49}$$

which does not involve λ.

After substituting (2.45) and (2.47) into (2.49), β is determined as $\beta = 0{\cdot}544$. From (2.44) λ may now be determined as

$$\lambda = 3{\cdot}03. \tag{2.50}$$

Because Φ' has no zero in its assumed form, it is thought that (2.50) should be an approximation to the lowest eigenvalue.

A more accurate value of the first eigenvalue and a tabulation of the first eigenfunction, and of the second eigenvalue and second eigenfunction, were generously obtained for the author by Mr A. Davey of the National Physical Laboratory by use of a Mercury electronic digital computer. The first two eigenvalues which he found are

$$\left.\begin{aligned}\lambda_1 &= 3{\cdot}0627 \pm 0{\cdot}0002,\\ \lambda_2 &= 5{\cdot}022 \;\; \pm 0{\cdot}002.\end{aligned}\right\} \tag{2.51}$$

The main computational difficulty was to ensure that the solutions with algebraic decay at infinity were excluded in favour of solutions with exponential decay: this criterion led to the eigenvalues (2.51).

The very accurate approximation to the first eigenvalue gained by use of the Pohlhausen method should be noticed.

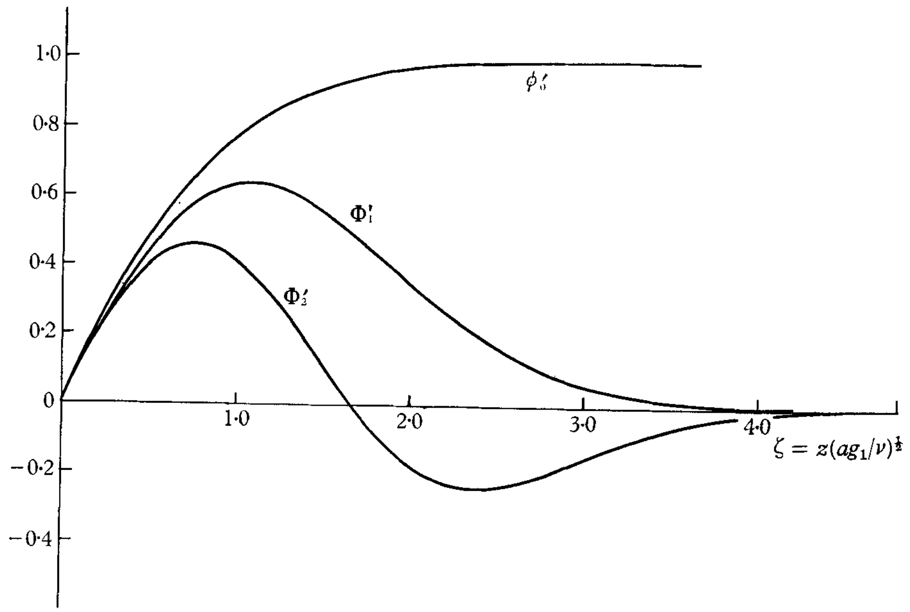

FIGURE 2. The derivatives of the first two eigenfunctions for two-dimensional stagnation flow.

The first two eigenfunctions and their first derivatives are listed in table 1 and the first derivatives are plotted in figure 2. One may see from figure 2 that Φ_1' exhibits no zero between its end points; this suggests that there is probably no lower real eigenvalue. Mr Davey was unable to find any real eigenvalue between $\lambda = 1{\cdot}5$ and $\lambda = \lambda_1$ but did not examine lower real or complex values. However, in view of the shape of Φ_1' and the earlier reasoning concerning complex values, we may assume that (2.51) gives the lowest two eigenvalues.

We have considered thus far the solution to be represented as a linear perturbation from the steady solution (see (2.11)). If the problem were truly linear, we could express the complete solution as

$$\phi(\zeta,\tau) = \phi_0(\zeta) + \sum_{j=1}^{\infty} A_j \Phi_j(\zeta, \lambda_j)\, e^{-\lambda_j \tau}, \tag{2.52}$$

where the first two eigensolutions are given above. The constants A_j would in principle be determined from the initial conditions. This expansion is precluded,

	$\lambda_1 = 3{\cdot}0627 \pm 0{\cdot}0002$			$\lambda_2 = 5{\cdot}022 \pm 0{\cdot}002$		
ζ	Φ_1	Φ_1'	Φ_1''	Φ_2	Φ_2'	Φ_2''
0·0	0·000	0·000	1·000	0·000	0·000	1·000
0·1	0·005	0·100	0·985	0·005	0·099	0·975
0·2	0·020	0·196	0·942	0·020	0·194	0·904
0·3	0·044	0·287	0·876	0·043	0·279	0·792
0·4	0·077	0·371	0·785	0·075	0·351	0·647
0·5	0·118	0·444	0·685	0·113	0·407	0·477
0·6	0·165	0·507	0·569	0·156	0·446	0·292
0·7	0·219	0·558	0·447	0·202	0·466	0·102
0·8	0·277	0·596	0·321	0·248	0·466	−0·084
0·9	0·338	0·622	0·197	0·294	0·449	−0·257
1·0	0·401	0·636	0·078	0·338	0·416	−0·408
1·1	0·464	0·638	−0·032	0·377	0·368	−0·533
1·2	0·528	0·630	−0·132	0·411	0·310	−0·626
1·3	0·590	0·612	−0·219	0·439	0·244	−0·685
1·4	0·650	0·586	−0·291	0·460	0·174	−0·711
1·5	0·707	0·554	−0·349	0·474	0·103	−0·705
1·6	0·761	0·517	−0·391	0·480	0·034	−0·670
1·7	0·811	0·477	−0·419	0·481	−0·030	−0·611
1·8	0·856	0·434	−0·433	0·475	−0·087	−0·534
1·9	0·897	0·390	−0·435	0·463	−0·136	−0·444
2·0	0·934	0·347	−0·426	0·448	−0·176	−0·347
2·1	0·967	0·305	−0·409	0·429	−0·205	−0·249
2·2	0·995	0·266	−0·385	0·407	−0·225	−0·154
2·3	1·020	0·229	−0·356	0·384	−0·236	−0·067
2·4	1·041	0·195	−0·324	0·360	−0·239	0·009
2·5	1·059	0·164	−0·290	0·336	−0·235	0·073
2·6	1·074	0·136	−0·256	0·313	−0·225	0·123
2·7	1·086	0·113	−0·223	0·291	−0·211	0·160
2·8	1·097	0·092	−0·192	0·271	−0·194	0·185
2·9	1·105	0·074	−0·162	0·253	−0·174	0·198
3·0	1·112	0·059	−0·136	0·236	−0·154	0·201
3·1	1·117	0·047	−0·112	0·222	−0·134	0·197
3·2	1·121	0·037	−0·092	0·209	−0·115	0·187
3·3	1·124	0·028	−0·074	0·199	−0·097	0·172
3·4	1·127	0·022	−0·059	0·190	−0·081	0·155
3·5	1·129	0·017	−0·046	0·182	−0·066	0·136
3·6	1·130	0·012	−0·036	0·177	−0·054	0·117
3·7	1·131	0·009	−0·028	0·172	−0·043	0·099
3·8	1·132	0·007	−0·021	0·168	−0·034	0·083
3·9	1·133	0·005	−0·016	0·165	−0·026	0·068
4·0	1·133	0·004	−0·012	0·163	−0·020	0·055
4·1	1·133	0·003	−0·009	0·161	−0·015	0·043
4·2	1·133	0·002	−0·006	0·159	−0·011	0·034
4·3	1·134	0·001	−0·005	0·159	−0·008	0·026
4·4	1·134	0·001	−0·003	0·158	−0·006	0·020
4·5	1·134	0·001	−0·002	0·157	−0·004	0·015
4·6	1·134	0·000	−0·002	0·157	−0·003	0·011
4·7	1·134	0·000	−0·001	0·157	−0·002	0·008
4·8	1·134	0·000	−0·001	0·156	−0·002	0·006
4·9	1·134	0·000	−0·001	0·156	−0·001	0·004
5·0	1·134	0·000	0·000	0·156	−0·001	0·003
5·1	1·134	0·000	0·000	0·156	0·000	0·002
5·2	1·134	0·000	0·000	0·156	0·000	0·001
5·3	1·134	0·000	0·000	0·156	0·000	0·001
5·4	1·134	0·000	0·000	0·156	0·000	0·001
5·5	1·134	0·000	0·000	0·156	0·000	0·000

TABLE 1. The first two eigenfunctions and their derivatives for two-dimensional stagnation flow.

however, by the non-linearity of the governing partial differential equation (2.10), and additional terms would be required in the expansion of $\phi(\zeta,\tau)$ in order to account for the interaction between the eigensolutions. The analytic determination of the constants for the non-linear problem is not clear. A simple numerical match could be attempted in the case of impulsive motion to the results of Goldstein & Rosenhead (1936), who developed a solution valid for small times by an expansion in powers of time. The problem of matching, however, will not be considered further in this paper.

An examination of the series (2.52) or its non-linear counterpart suggests that they are convergent for $\tau \to \infty$ with ζ fixed, and in this limit the solution described in this paper certainly seems to be valid. If, however, we consider the limit $\zeta \to \infty$ with τ fixed it is not clear that the series are convergent; this can be seen for the series (2.52) from the asymptotic behaviour (2.31) for each possible value of λ. In order to decide about convergence it would be necessary to solve the initial-value problem and thus to determine the constant coefficients in the series. In the absence of such information, however, it is suggested that the eigen-solution series is probably an 'unsuitable' expansion of the actual solution (in the sense, by way of analogy, that $\exp(-\frac{1}{2}x^2+\epsilon x^2)$ can be expanded as

$$\sum_{n=0}^{\infty} \epsilon^n x^{2n} e^{-\frac{1}{2}x^2}/n!,$$

which gives incorrect behaviour at infinity if only a finite number of terms is known).

The solution obtained here satisfies the specified boundary conditions at infinity, and it seems unlikely that an incorrect exponential behaviour there greatly affects the solution for finite values of ζ. The difficulty discussed has arisen partly because the type of linearization used to derive equation (2.13) is invalid when $\zeta \to \infty$ for τ fixed, and partly because of the chosen method of solution of the linearized problem (2.13) in terms of eigensolutions derived after the separability assumption (2.15).

The above results may be of use in connexion with certain three-dimensional problems. For instance, consider the flow near the line of stagnation points of an infinite cylinder whose axis is normal to the free stream. When the cylinder undergoes unsteady motion of the above type in both the free stream and axial directions, the axial flow possesses additional eigensolutions but the chordwise flow is still governed by the above result.

3. Axially symmetric stagnation flow

The analysis in this case is quite similar to the preceding development, and therefore only the major steps are indicated. If r and z denote, respectively, the radial and axial directions, and v_r and v_z denote the velocity components in those directions, the momentum equation in the radial direction for the case of axial symmetry is

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_r}{\partial z^2}\right). \tag{3.1}$$

The boundary conditions are

$$v_r = v_z = 0 \quad \text{at} \quad z = 0, \qquad v_r \to V_r(r,t) \quad \text{as} \quad z \to \infty. \tag{3.2}$$

The following substitutions are made

$$\left.\begin{aligned} \bar{\tau} = at, \quad \zeta = z[ag(\bar{\tau})/\nu]^{\frac{1}{2}}, \quad V_r(r,t) = \arg(\bar{\tau}), \\ v_r = \arg(\bar{\tau})\,\phi'(\zeta,\bar{\tau}), \quad v_z = -2[a\nu g(\bar{\tau})]^{\frac{1}{2}}\,\phi(\zeta,\bar{\tau}), \\ p = -\tfrac{1}{2}\rho[a^2r^2g^2(\bar{\tau}) + a\nu G(\zeta,\bar{\tau})] - \tfrac{1}{2}\rho a^2 r^2 \dot{g}(\bar{\tau}). \end{aligned}\right\} \tag{3.3}$$

	$\lambda_1 = 3{\cdot}9828 \pm 0{\cdot}0003$			$\lambda_2 = 7{\cdot}86 \pm 0{\cdot}01$		
ζ	Φ_1	Φ_1'	Φ_1''	Φ_2	Φ_2'	Φ_2''
0·0	0·000	0·000	1·000	0·000	0·000	1·000
0·1	0·005	0·099	0·980	0·005	0·099	0·961
0·2	0·020	0·195	0·922	0·019	0·190	0·847
0·3	0·044	0·282	0·827	0·042	0·266	0·669
0·4	0·076	0·359	0·702	0·072	0·322	0·441
0·5	0·115	0·422	0·553	0·106	0·353	0·185
0·6	0·160	0·469	0·389	0·142	0·359	−0·076
0·7	0·208	0·500	0·220	0·177	0·339	−0·318
0·8	0·259	0·513	0·054	0·209	0·296	−0·521
0·9	0·311	0·511	−0·100	0·236	0·236	−0·668
1·0	0·361	0·494	−0·232	0·256	0·165	−0·750
1·1	0·409	0·465	−0·339	0·268	0·089	−0·766
1·2	0·454	0·427	−0·417	0·273	0·014	−0·720
1·3	0·494	0·383	−0·465	0·271	−0·054	−0·625
1·4	0·530	0·335	−0·484	0·263	−0·110	−0·496
1·5	0·561	0·287	−0·479	0·250	−0·152	−0·349
1·6	0·588	0·240	−0·454	0·233	−0·180	−0·201
1·7	0·609	0·197	−0·415	0·214	−0·193	−0·066
1·8	0·627	0·157	−0·366	0·195	−0·194	0·047
1·9	0·641	0·123	−0·314	0·176	−0·185	0·133
2·0	0·652	0·095	−0·261	0·158	−0·168	0·190
2·1	0·660	0·071	−0·211	0·142	−0·147	0·220
2·2	0·666	0·052	−0·166	0·129	−0·125	0·228
2·3	0·671	0·038	−0·128	0·117	−0·102	0·219
2·4	0·674	0·027	−0·096	0·108	−0·081	0·198
2·5	0·676	0·018	−0·070	0·110	−0·063	0·170
2·6	0·678	0·012	−0·050	0·096	−0·047	0·141
2·7	0·679	0·008	−0·035	0·092	−0·035	0·112
2·8	0·679	0·005	−0·024	0·089	−0·025	0·086
2·9	0·680	0·003	−0·016	0·086	−0·017	0·064
3·0	0·680	0·002	−0·010	0·085	−0·012	0·047
3·1	0·680	0·001	−0·007	0·084	−0·008	0·033
3·2	0·680	0·001	−0·004	0·083	−0·005	0·022
3·3	0·680	0·000	−0·002	0·083	−0·003	0·015
3·4	0·681	0·000	−0·001	0·083	−0·002	0·010
3·5	0·681	0·000	−0·001	0·083	−0·001	0·006
3·6	0·681	0·000	0·000	0·082	−0·001	0·004
3·7	0·681	0·000	0·000	0·082	0·000	0·002
3·8	0·681	0·000	0·000	0·082	0·000	0·001
3·9	0·681	0·000	0·000	0·082	0·000	0·001
4·0	0·681	0·000	0·000	0·082	0·000	0·000

TABLE 2. The first two eigenfunctions and their derivatives for axially symmetric stagnation flow.

The momentum equation (3.1) becomes

$$\dot{g}\phi' + g\dot{\phi}' + \tfrac{1}{2}\zeta\dot{g}\phi'' + (g\phi')^2 - 2g^2\phi\phi'' = g^2 + \dot{g} + g^2\phi''', \tag{3.4}$$

with the boundary conditions

$$\phi(0,\bar{\tau}) = \phi'(0,\bar{\tau}) = 0, \quad \phi'(\infty,\bar{\tau}) = 1. \tag{3.5}$$

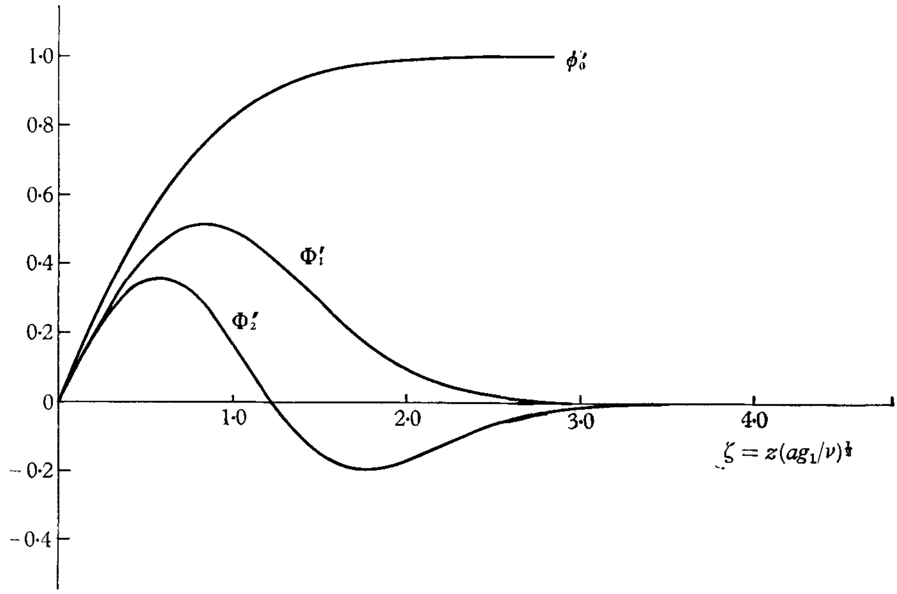

FIGURE 3. The derivatives of the first two eigenfunctions for axially symmetrical stagnation flow.

We again consider the case when $g(\bar{\tau})$ reaches a steady value and look for a solution of (3.4) which represents a perturbation on the axially symmetric steady-state solution, $\phi_0(\zeta)$. In complete analogy to (2.17) and the steps preceding it, the equation for the perturbation is

$$\Phi''' + 2\phi_0\Phi'' - 2\phi_0'\Phi' + 2\phi_0''\Phi + \lambda\Phi' = 0. \tag{3.6}$$

The order of equation (3.6) may be reduced by use of the substitution

$$\Phi(\zeta) = (\phi_0' - \tfrac{1}{2}\lambda)\,\theta(\zeta). \tag{3.7}$$

One may then prove the non-existence of complex eigenvalues with negative real parts and the existence of a discrete set of eigenvalues for sufficiently large λ by the methods of § 2.

A Pohlhausen method may again be applied to estimate the first eigenvalue. The integral form of (3.6) is

$$\lambda + 2 = \left\{\Phi''(0) + 6\int_0^\infty \phi_0'\,\Phi'\,d\zeta\right\}\Big/\Phi(\infty). \tag{3.8}$$

With reference to (2.45), corresponding values of the constants for this case are $\gamma = 2{\cdot}0826$, $b = 2{\cdot}8532$, $c = -1{\cdot}5412$, and, with reference to (2.47), $\beta = 0{\cdot}6977$. The result is

$$\lambda = 4{\cdot}01. \tag{3.9}$$

The first two eigenvalues, obtained by Mr Davey on the Mercury digital computer, are

$$\left.\begin{aligned} \lambda_1 &= 3{\cdot}9828 \pm 0{\cdot}0003, \\ \lambda_2 &= 7{\cdot}86 \quad \pm 0{\cdot}01. \end{aligned}\right\} \tag{3.10}$$

As before algebraic decay of the solution at infinity is excluded in favour of exponential decay. The corresponding eigenfunctions are given in table 2, and their first derivatives are shown in figure 3. One must, of course, make the same qualifications concerning these eigensolutions as were made in the two-dimensional case.

The agreement between the Pohlhausen result for the first eigenvalue and the computer calculation is again excellent.

4. Impulsive motion of a flat plate normal to its leading edge

Discussion of the above problem was begun with the knowledge that Professor C. C. Lin had demonstrated sometime ago in unpublished work that related behaviour was impossible for impulsive Blasius flow. An independent proof is offered here.

Consider the boundary-layer equations for impulsive Blasius flow

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} &= 0. \end{aligned}\right\} \tag{4.1}$$

The boundary conditions are

$$\left.\begin{aligned} &u = w = 0 \quad \text{for} \quad z = 0, \quad x \geqslant 0, \quad \text{all } t, \\ &u \to U \quad \text{for} \quad z \to \infty, \quad \text{all } x, \quad \text{all } t, \\ &t \geqslant 0, \quad U = U_0; \qquad t < 0, \quad u = U = 0. \end{aligned}\right\} \tag{4.2}$$

The following substitutions are made

$$\left.\begin{aligned} &n = z(U_0/\nu x)^{\frac{1}{2}}, \quad \tau = U_0 t/x, \\ &\psi = (U_0 \nu x)^{\frac{1}{2}}\,\phi(n,\tau), \\ &u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \end{aligned}\right\} \tag{4.3}$$

If $\phi_0(n)$ represents the steady-state solution, consider a perturbation on ϕ_0 which is assumed to be separable

$$\phi(n,\tau) = \phi_0(n)+\epsilon\phi_1(n)\,A(\tau). \tag{4.4}$$

The equation for the perturbation is

$$A\phi_1''' + \tfrac{1}{2}A\phi_0\phi_1'' + \tfrac{1}{2}A\phi_0''\phi_1 + \tau\dot{A}[\phi_0'\phi_1' - \phi_0''\phi_1] - \dot{A}\phi_1' = 0, \tag{4.5}$$

$$\phi_1(0) = \phi_1'(0) = \phi_1'(\infty) = 0. \tag{4.6}$$

If $A(\tau)$ were assumed to be an exponential function of τ, i.e. $A(\tau) = e^{\omega\tau}$ where ω is a constant, the term in brackets in (4.5) would be predominant at large times. If only this term were solved for a first approximation, a non-trivial solution could not satisfy the boundary conditions (4.6). Hence $A(\tau)$ is assumed in the form

$$A(\tau) = \tau^{\omega}$$

so that $\tau\dot{A} \sim A$. Then, for large τ, the last term in (4.5) is of smaller order, and (4.5) reduces to

$$\phi_1''' + \tfrac{1}{2}\phi_0\phi_1'' + \tfrac{1}{2}\phi_0''\phi_1 + \omega(\phi_0'\phi_1' - \phi_0''\phi_1) = 0. \tag{4.7}$$

The order of (4.7) may be reduced by the substitution

$$\phi_1 = \phi_0'\theta, \tag{4.8}$$

which gives

$$\theta''' + \{3(\phi_0''/\phi_0') + \tfrac{1}{2}\phi_0\}\theta'' + \{(\phi_0'''/\phi_0') + \omega\phi_0'\}\theta' = 0, \tag{4.9}$$

with the boundary conditions

$$\theta'(\infty) = 0, \quad \theta'(0) = \text{finite, non-zero.} \tag{4.10}$$

Assuming ω to be real, we multiply (4.9) by $(\phi_0')^3 E\theta'$, where

$$E = \exp\left\{\frac{1}{2}\int_0^{\eta} \phi_0 dn\right\},$$

and integrate from $n = 0$ to ∞. The resulting integral may be written as

$$\omega = \frac{\displaystyle\int_0^{\infty} E\phi_0'[(\phi_0'\theta'')^2 - \phi_0'''\phi_0'(\theta')^2]\,dn}{\displaystyle\int_0^{\infty} (\phi_0')^4 E(\theta')^2\,dn}. \tag{4.11}$$

Because $\phi_0' \geqslant 0$ and $\phi''' \leqslant 0$, ω must be positive. Thus the perturbations become infinite as $\tau \to \infty$, and the assumption of linearization is invalidated. It may also be shown by an analysis similar to that in § 2.1 that there are no possible complex values of ω.

Since one expects the solution to approach the steady Blasius result, it appears that one cannot study the approach to the steady state by means of a linearized perturbation of the separable type (4.4). This is in distinct contrast to the case of stagnation point flow discussed earlier. The result generalizes Stewartson's (1951) statement that $\phi(n,\tau)$ cannot be expanded as an inverse power series in τ where the coefficient of each τ^n is only a function of n.

After completing this paper, the author has become aware of the work by Lam & Rott (1960), who gave a related eigenfunction analysis for the boundary-layer equation.

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